Random Features Approximation for Fast Data-Driven Control

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Abstract

The goal of data-driven nonlinear control problems is to guarantee stability or safety of an unknown system. We consider a method based on Control Certificate Functions (CCFs) that uses Gaussian Process (GP) regression to learn unknown quantities for control affine dynamics. To make practical use of the data-driven GP controller, it is necessary to compute posterior estimates in real-time feedback systems. As computing the GP estimator is expensive, We propose a suitable RF method to speed up computations. We further provide a probabilistic error analysis to propose a robust second order cone program based controller.

Control Certificate Functions

Our goal is to design a controller that guarantees properties like safety and stability for a control affine dynamical system: $\dot{x} = f(x) + g(x)u$. We use an approach based on control certificate functions, a family of functions including control Lyapunov and Barrier functions. They are used as a constraint in a min-norm quadratic program (QP) to synthesize controllers that guarantee our desired properties:

$$u^*(x) = \underset{u\in\mathbb{R}^m}{\arg\min}\, \|u\|_2^2 \qquad \textbf{(CCF-QP)}$$
$$\text{s.t.} \quad \underbrace{\nabla C(x)^\top (f(x)+g(x)u)}_{:=\dot{C}(x,u)} + \alpha(C(x)) \le 0.$$

However, to implement this QP, is it necessary to know the dynamics f and g. Since we don't have access to those, we estimate $\dot{C}(x,u)$. We consider a setting where the model is unknown but that a valid CCF function C and α for the true plant is known.

Random Features Approximation

Computing the GP estimator can become prohibitively expensive for large datasets, which is an issue since speed is critical in real time control systems. We introduce a random feature approximation of the affine compound kernel to speed up training and prediction time.

$$\psi(x) := \sqrt{\frac{2}{D}}\left[sin(\omega_1^\top x)\ cos(\omega_1^\top x) \ldots sin(\omega_{D/2}^\top x)\ cos(\omega_{D/2}^\top x)\right]^\top$$

where $\omega_i \overset{iid}{\sim} p(\omega)$ and $p(\omega)$ is the Fourier transform of the kernel (which by Böchner's theorem, is a valid density function). Thus we have that $k(x-x') = \mathbb{E}_\omega[\psi_\omega(x)^\top \psi_\omega(x')]$. Sutherland and Schneider [2] show that with probability $1-\delta_2$, we have ϵ pointwise error bounds on the error of this estimation, for some appropriate D. We can compute a random feature approximation in $O(nD^2)$ time and $O(nD)$ memory, which is computationally attractive if $D < n$.

Data-driven Controller with GP Regression

We develop a data-driven control synthesis method which uses data of the form $\{(x_i, \dot{C}(x_i,u_i)\}_{i=1}^n$ We can approximately measure $\dot{C}(x,u)$ using finite differencing methods on sampled trajectories from the true system. However, to ensure the the true system certifies the desired property, we must account for prediction errors. Therefore, we use GP regression. Under GP assumptions, given a set of finite measurements of features and labels, at a query point, a posterior distribution can be derived. We note that $\dot{C}(x,u)$ is affine in u.For convenience of notation, denote $s = [x^\top\ \ 1\ \ u^\top]^\top$, and define $\bar{\mathcal{X}} = \mathcal{X} \times \{1\} \times \mathbb{R}^m$.

Definition: Define the Affine Dot Product (ADP) compound kernel of $(m+1)$ individual kernels $k_i : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ as $k_c : \bar{\mathcal{X}}\times\bar{\mathcal{X}} \to \mathbb{R}$ given by

$$k_c(s,s') := \begin{bmatrix}1\\u\end{bmatrix}^\top \begin{bmatrix}k_1(x,x') & & \\ & \ddots & \\ & & k_{m+1}(x,x')\end{bmatrix} \begin{bmatrix}1\\u'\end{bmatrix}$$

Prior work [1] uses that with a probability of at least $1-\delta_1$, we have:

$$\dot{C}_x(u) \le \mu_x(u) + \beta\sigma_x(u)\ .$$

As first proposed by [1], we construct a Second-Order Cone Program (SOCP) which defines a data-driven min-norm stabilizing feedback control law $u^* \colon \mathbb{R}^n \to \mathbb{R}^m$:

$$u^*(x) = \underset{u\in\mathbb{R}^m}{\arg\min}\, \|u\|_2^2 \qquad \textbf{(GP-CCF-SOCP)}$$
$$\text{s.t.} \quad \mu_x(u) + \beta\sigma_x(u) + \alpha(C(x)) \le 0$$

Results on Random Features GP control

We use a random features approximation to estimate the ADP compound kernel. We approximate only the state-dependent portion, defining the random feaures as:

$$\varphi(s_i)^\top = [\psi_1(x_i)^\top\ u_i^1\psi_1(x_i)^\top \ldots u_i^m\psi_{m+1}(x_i)^\top] = (\phi(x_i)\begin{bmatrix}1\\u_{x_i}\end{bmatrix})^\top$$

where $\phi(x) := \text{blkdiag}[\psi_1(x); ...; \psi_{m+1}(x)]$. Let $\Phi \in \mathbb{C}^{n\times D(m+1)}$ be the matrix whose i^{th} row is $\varphi(s_i)^\top$. Then the posterior mean and covariance can be approximated by

$$\hat{\mu}_x(u) = \varphi(s)^\top(\Phi^\top\Phi + \lambda_n I)^{-1}\Phi^\top z$$
$$\hat{\sigma}_x(u) = \lambda_n \varphi(s)^\top(\Phi^\top\Phi + \lambda_n I)^{-1}\varphi(s).$$

Where z is the vector containing the output measurements. Note that the time complexity of computing μ is $O(n(m+1)^2D^2)$ as opposed to $O(n^3)$. For the purposes of robustly guaranteeing feasibility of (CCF-QP), it is necessary to track how the approximation error accumulates in our computation of the posterior.

Error Bounds

We provide an error analysis on the approximate mean and variance, which allows for the design of a quadratic SOCP.

Proposition: under reasonable assumptions, with a probability of $1-(\delta_1+\delta_2)$ we have:

$$|\dot{C}(s) - \hat{\mu}_x(u)| \le \beta\hat{\sigma}_x(u) + \epsilon(\nu\|u_x\| + \iota\|u_x\|^2 + \Delta)$$

where ν, ι and δ depend on information from the training data, number of training points, β, λ_n, and the measurement noise. Finally, we propose a fast and robust convex optimization based min-norm controller using the error bounds, that is both computationally efficient and robust:

RF-CCF-SOCP

$$u^*(x) = \underset{u\in\mathbb{R}^m}{\arg\min}\|u\|_2^2$$
$$\text{s.t.} \quad \hat{\mu}_x(u) + \beta\hat{\sigma}_x(u) + \epsilon(\nu\|u_x\| + \iota\|u_x\|^2 + \Delta) + \alpha(C(x)) \le 0$$

Experiments

We present preliminary experiments of the random features approach for an adaptive cruise control system [1]. There is is a reduction of 85% in training time when using RF approximation vs ADP compound kernel. We see that the RF predictions are nearly identical, although sometimes it underestimates the variance compared with the full ADP kernel, which highlights the importance of incorporating the error analysis presented.

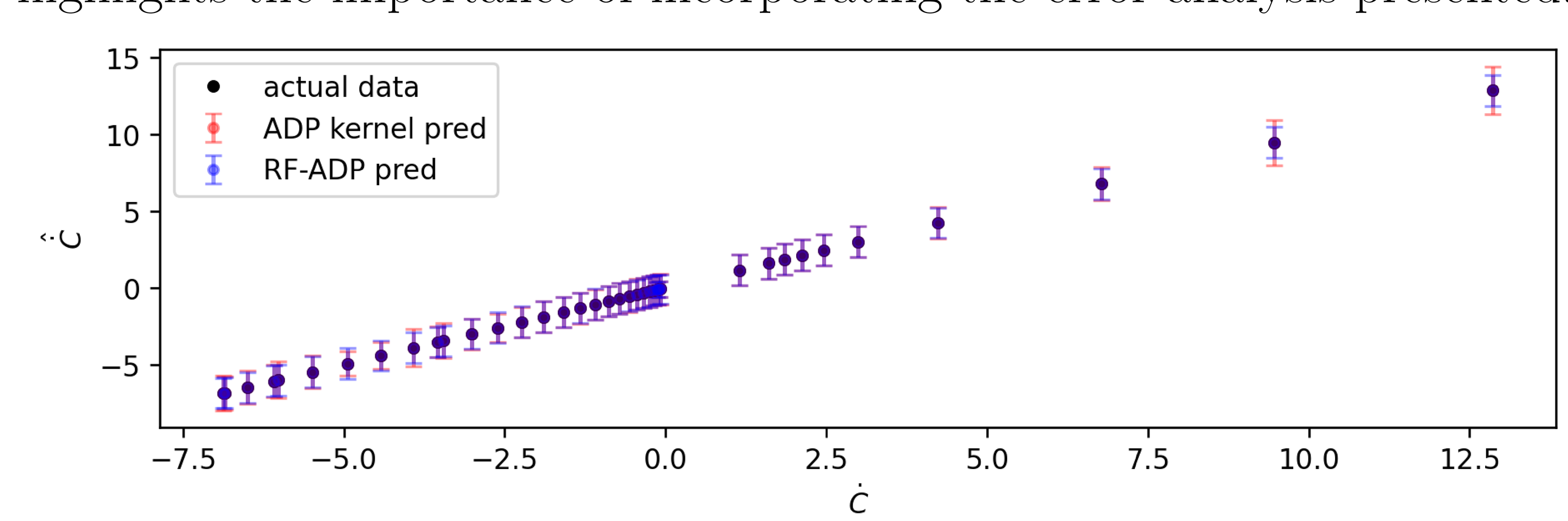

Figure: Predicted mean (dot) and variance (bars) for $\dot{C}$ in the final episode of training data.

[1] F. Castañeda, J. J. Choi, B. Zhang, C. J. Tomlin, and K. Sreenath. Pointwise feasibility of gaussian process-based safety-critical control under model uncertainty. *In 2021 60th IEEE Conference on Decision and Control (CDC)*, pages 6762–6769, 2021.

[2] D. Sutherland and J. Schneider. On the error of random fourier features. *Uncertainty in Artificial Intelligence - Proceedings of the 31st Conference, UAI 2015*, 06 2015.